Functions of random variables, Expectation

Putting a value on random variables

Michael Psenka

What we aim to model

E.g. 5 coin flips.

$\Omega = \{H,T\} \times \{H,T\} \times \dots \times \{H,T\}$

$(H,T,T,H,T) \in \Omega$

$\mathbb{P}(\#\text{heads} = k)$?

$(H,T,T,H,T) \xrightarrow{X(\omega)} \sum_{i=1}^{5} \mathbb{1}(\omega_i = H)$

"sum up # heads"

$:= \mathbb{P}(X(\omega) = 3)$

$\mathbb{P}(\#\text{heads} = 3)$?

"#heads = 3" $= \{\omega \in \Omega : X(\omega) = 3\}$

$= \boxed{\frac{1}{2^5}\binom{5}{3}}$

"Random variable": notes

Def A "random variable" is a function

$X : \Omega \to \mathbb{R}$

$\Omega = \{H, T\}^{\times 5}$

$\mathbb{P} =$ "unif. over Ω"

X, "number of heads" $\longrightarrow$

$\Omega' = \{0, 1, 2, 3, 4, 5\} \subseteq \mathbb{R}$

$\mathbb{P}' = \{ \frac{1}{2^5}, \binom{5}{1} \frac{1}{2^5}, \binom{5}{2} \frac{1}{2^5}, \dots \}$

$X : \{H, T\}^{\times 5} \to \mathbb{R}$

$(H, T, T, H, T) \qquad 2$

$\binom{5}{k} \frac{1}{2^5}$

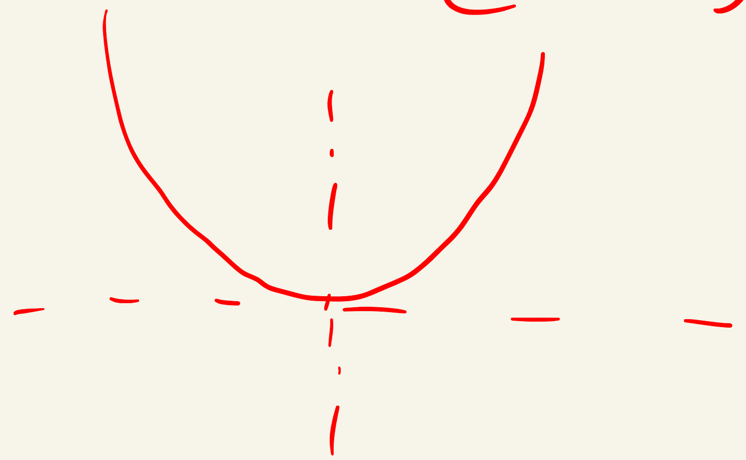
Functions induce random variables

Numeric random variables

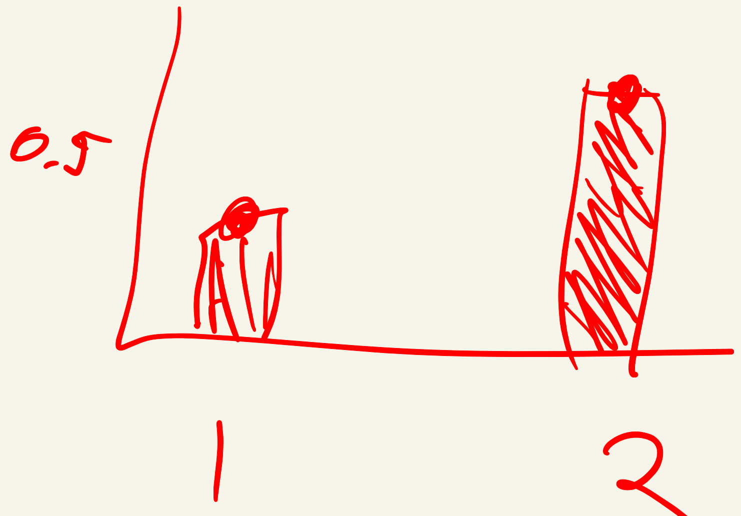
A random variable (rec) is numeric if $\Omega \subset \mathbb{R}$

Distribution (graphical)

for function $f: \mathbb{R} \to \mathbb{R}$, the graph of f is $\{(x, f(x))\}$



The graph of a distribution is the set $\{(\omega, P(\omega))\}_{\omega \in \Omega}$



e.g. coin, biased towards "tails" (2)

Examples: Bernoulli

$X = (\Omega, \mathbb{P})$

$\Omega = \{0, 1\}$, $\mathbb{P}(\{0\}) = p$ $\mathbb{P}(X=0)$ $\mathbb{P}(X=k)$

$\mathbb{P}(\{1\}) = 1-p$ $\mathbb{P}(X=1)$

e.g. coin, $p=0.5$, weighted coin, $p=0.2$

Examples: Binomial

"n Bernoulli r.v., all w/ probability p"

$\Omega_1 = \{0,1\} \times \{0,1\} \times \dots \times \{0,1\}$ (n copies), dist P'

$\Omega_2 = f(\Omega_1)$, $f :=$ "sum of indices" $= \sum_{i=1}^{n} \omega_i$

$\mathbb{P}(X=k) = \sum_{a \in E} P'(a)$, $E := \{a \mid \text{exactly } k \text{ 1's}\}$

$= \sum_{a \in E} p^k (1-p)^{n-k} = \boxed{\binom{n}{k} p^k (1-p)^{n-k}}$

(Numerical) joint distributions

	P	N
I	0.8	0.05
Not I	0.05	0.1

Y \ X	0	1
0	0.8	0.05
1	0.05	0.1

$P(X=0, Y=0)$

$P(X=1, Y=0)$

Two r.v. X, Y, they have a joint distribution $P(X \times Y)$

E.g. 1, two independent coins

0.25	0.25
0.25	0.25

E.g. 2 "entangled coins"

0	0.5
0.5	0

Marginal distributions

X, Y $\quad\quad \mathbb{P}(X=k, Y=\ell)$

$$\mathbb{P}(X=k) = \mathbb{P}(X=k \mid Y=1)\,\mathbb{P}(Y=1) + \mathbb{P}(X=k \mid Y=2)\,\mathbb{P}(Y=2) + \ldots$$

$$= \mathbb{P}(X=k, Y=1) + \mathbb{P}(X=k, Y=2) + \ldots$$

$$= \sum_{i \in \Omega_Y} \mathbb{P}(X=k, Y=i)$$

Independence

$X, Y \qquad \mathbb{P}(X=k, Y=l)$

Def: X, Y indep if $\forall\, k, l$

$\underline{\mathbb{P}(X=k, Y=l) = \mathbb{P}(X=k)\mathbb{P}(Y=l)}$

$\Rightarrow \mathbb{P}(X \in E, Y \in F) = \mathbb{P}(X \in E)\mathbb{P}(Y \in F)$

$\mathbb{P}(X=k, Y=l) \Rightarrow \mathbb{P}(X \in E, Y=l) \Rightarrow \mathbb{P}(X \in E, Y \in F)$

Expectation

Definition. *The expectation of a numeric random variable* $X = (\Omega, \mathbb{P})$ *is given by the following:*

$$\mathrm{E}[X] := \sum_{x \in \Omega} x \mathbb{P}(X = x).$$

e.g. Bernoulli w/ $p = 0.5$, $E[X] = 0 \cdot 0.5 + 1 \cdot 0.5 = 0.5$

Binomial, $E[X] = \sum \underbrace{k}_{\text{"}x\text{"}} \underbrace{(p)^k (1-p)^{n-k} \binom{n}{k}}_{\mathbb{P}(X=x)}$

Linearity of expectation

Theorem. *Let X, Y be two (numeric) random variables, and their joint distribution given by $\mathbb{P}(X = x, Y = y)$. The expectation is a "linear operator": that is,*

1. $\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y]$,
2. $\mathrm{E}[cX] = c\mathrm{E}[X]$ for any fixed $c \in \mathbb{R}$.

Linearity of expectation

Pf Writing out $\mathbb{E}[X+Y] = \sum (x+y)\,\mathbb{P}(X=x, Y=y)$

$$= \sum_{x\in\Omega_X}\sum_{y\in\Omega_Y} (x+y)\,\mathbb{P}(X=x, Y=y)$$

$$= \sum_x\sum_y x\,\mathbb{P}(X=x, Y=y) + \sum_y\sum_x y\,\mathbb{P}(X=x, Y=y)$$

$$= \sum_x x\Big(\sum_y \mathbb{P}(X=x, Y=y)\Big) + \sum_y y\Big(\sum_x \mathbb{P}(X=x, Y=y)\Big)$$

$$= \sum_x x\,\mathbb{P}(X=x) + \sum_y y\,\mathbb{P}(Y=y) = \mathbb{E}[X] + \mathbb{E}[Y].$$

Product of independent expectations

Theorem. *Let X, Y be two (numeric) random variables, and their joint distribution given by $\mathbb{P}(X = x, Y = y)$. If X, Y are independent, then expectation factors out their product:*

$$\mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y].$$

Product of independent expectations

$$\mathbb{E}[XY] = \sum_{y} \sum_{y} xy\,\mathbb{P}(X=x, Y=y)$$

$$= \sum_{x} \sum_{y} xy\,\mathbb{P}(X=x)\,\mathbb{P}(Y=y)$$

$$= \sum_{x} x\,\mathbb{P}(X=x) \left(\sum_{y} y\,\mathbb{P}(Y=y) \right)$$

$$= \mathbb{E}[Y] \left(\sum_{x} x\,\mathbb{P}(X=x) \right) = \mathbb{E}[Y]\,\mathbb{E}[X]$$